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Buckling of a *von Kármán* Plate Adhesively Connected to a Rigid Support Allowing for Delamination: Existence and Multiplicity Results

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Abstract. The present paper deals with an eigenvalue problem for a hemivariational inequality, arising in the study of a mechanical problem: the buckling of a *von Kármán* plate adhesively connected to a rigid support with delamination effects. For this eigenvalue problem an existence result is obtained by applying a critical point method suitable for nonconvex nonsmooth functions. Further, a result concerning the multiplicity of solutions is proved. The mechanical interpretation of these results is briefly discussed.

Key words: Critical points; Eigenvalue problem; Hemivariational inequality

1. Introduction

In the present paper a static problem of nonsmooth mechanics concerning a special category of plates, the *von Kármán* plates, is discussed. The *von Kármán* plates are thin plates undergoing moderate large deflections. The theory used for their stability analysis is the *von Kármán* plate theory. In the mechanical model studied, a *von Kármán* plate is adhesively connected to a rigid support. The adhesive contact law is nonmonotone and can be derived by a nonconvex nonsmooth potential, called superpotential. This law leads the variational formulation of the problem to a hemivariational inequality if classical boundary conditions are assumed to hold. The theory of hemivariational inequalities has been introduced and developed by P.D. Panagiotopoulos (see Naniewicz and Panagiotopoulos, 1995; Panagiotopoulos, 1993, 1985b). This theory generalizes the classical variational inequalities, for problems involving nonconvex nonsmooth functions. The variational approach in the study of hemivariational inequalities permits the search of the qualitative behavior of their solutions (the most recent work is presented in Motreanu and Panagiotopoulos (1998).

The plate studied is subjected to buckling forces in its plane. Due to the parametric expression of the buckling forces, the buckling problem of the adhesively supported *von Kármán* plate is an eigenvalue hemivariational inequality. This eigenvalue hemivariational inequality on the sphere, is the variational formulation of the buckling problem for prescribed cost or weight, or for a given energy consumption. Given the relation of the theory of hemivariational inequalities with nonconvex nonsmooth optimization, the considered eigenvalue problem can be equivalently formulated in a superpotential form. Its solution(s) characterize the equilibrium state(s) of the plate. One should remind that the mechanical problem, due to nonconvexity, admits in general multiple solutions.

In the mathematical study, the first concern is to search for local critical points, which correspond to the solutions of the mechanical problem. Thus an existence theorem is proved by using the classical critical point method. Further a multiplicity result is given for the solutions of the eigenvalue hemivariational inequality on the sphere, following Theorem 3.2 of Chang (1981).

Existence and approximation results for related plate problems have been derived in Panagiotopoulos (1985a, 1989), Panagiotopoulos and Stavroulakis (1988, 1990). Related mathematical problems for eigenvalue problems in hemivariational inequalities have been studied in Bocea, Motreanu and Panagiotopoulos (to appear), Motreanu and Panagiotopoulos (1995a, 1995b, 1996, 1997).

2. The mechanical study

Let us consider a *von Kármán* plate, of constant thickness h, adhesively connected to a rigid support. In the undeformed state the middle surface of the plate occupies an open bounded and connected subset Ω of R^2, referred to a fixed right-handed Cartesian coordinate system $Ox_1x_2x_3$. The plate lies in the Ox_1x_2 plane. The boundary Γ of the plate is assumed to be appropriately regular (in general, a Lipschitz boundary $C^{0,1}$ is sufficient). Let also the binding material occupy a subset Ω' such that $\Omega' \subset \Omega$ and $\overline{\Omega'} \cap \Gamma = \emptyset$.

We denote by $\zeta(x)$ the vertical deflection of the point $x \in \Omega$ and by $f = (0, 0, f_3(x))$ the distributed vertical load. Further, let $u = \{u_1, u_2\}$ the inplane displacement of the plate.

The plate obeys the *von Kármán* plate theory, which gives rise to the following system of differential equations

$$K\,\Delta\Delta\zeta - h(\sigma_{a\beta}\zeta_{,\beta})_{,a} = f\,, \tag{2.1}$$

and

$$\sigma_{a\beta,\beta} = 0\,, \tag{2.2}$$

with

$$\sigma_{a\beta} = C_{a\beta\gamma\delta}\left(\varepsilon_{\gamma\delta}(u) + \frac{1}{2}\,\zeta_{,\gamma}\zeta_{,\delta}\right) \tag{2.3}$$

$$\varepsilon_{a\beta}(u) = \frac{1}{2}\,(u_{a,\beta} + u_{\beta,a})\,. \tag{2.4}$$

Here the subscripts $a, \beta, \gamma, \delta = 1, 2$ correspond to the coordinate directions; $\{\sigma_{a\beta}\}$,

$\{\varepsilon_{a\beta}\}$ and $\{C_{a\beta\gamma\sigma}\}$ denote the stress, strain and elasticity tensors in the plane of the plate. The components of C are elements of $L^{\infty}(\Omega)$ and have the usual symmetry and ellipticity properties. Moreover, $K = Eh^3/12(1-\nu^2)$ is the bending rigidity of the plate with E the modulus of elasticity and ν the Poisson ratio. The plate thickness is denoted by h. For simplicity we consider the plate to be isotropic and homogeneous. In the simplified mechanical model studied, delamination effects are caused by the interlaminar normal stress f. For the mathematical calculus, f is split into $\bar{f}$, which describes the action of the adhesive and $\bar{\bar{f}} \in L^2(\Omega)$, which represents the external loading applied on the plate:

$$f = \bar{f} + \bar{\bar{f}} \text{ in } \Omega\,. \tag{2.5}$$

The mechanical behavior of the adhesive material as well as cracking and crushing effects, are given by a phenomenological law, connecting $\bar{f}$ with the corresponding deflection of the plate:

$$-\bar{f} \in \tilde{\beta}(\zeta) \text{ in } \Omega'\,, \tag{2.6}$$

where $\tilde{\beta}$ is a multivalued nonmonotone function defined as in Panagiotopoulos (1993), Equation (1.2.53). Its graph is made by filling in the jumps in the graph of a function $\beta \in L^{\infty}_{\mathrm{loc}}(\mathrm{R})$.

The following relation complete in a natural way the definition of $\bar{f}$:

$$\bar{f} = 0 \text{ in } \Omega - \Omega'\,. \tag{2.7}$$

It can be proved (cf. Chang, 1981) that a locally Lipschitz function j exists, with

$$j(\xi) = \int_0^{\xi} \beta(\xi_1)\,\mathrm{d}\xi_1\,, \tag{2.8}$$

such that $\bar{\partial} j(\xi) \subset \tilde{\beta}(\xi)$. If $\beta(\xi\pm)$ exists for every $\xi \in R$, then (cf. Panagiotopoulos, 1993, Section 1.2)

$$\hat{\beta}(\xi) = \bar{\partial} j(\xi)\,, \tag{2.9}$$

where $\bar{\partial}$ is the generalized gradient of Clarke.

From relation (2.1) it is obvious that only adhesive forces in the normal to the plate direction are considered in this model. Slip between the lower surface of the plate and the rigid support is considered to occur without resistance force (cf. a frictionless contact boundary). A more complicated model with skin effects has been proposed and studied in Panagiotopoulos (1989).

We assume that the following boundary conditions hold on the plate boundary

$$\zeta = 0 \text{ on } \Gamma\,. \tag{2.10}$$

For the in-plane action of the plate we assume the following parametrized boundary loading conditions

$$\sigma_{a\beta} n_{\beta} = \lambda g_a,\ a = 1, 2 \text{ on } \Gamma\,, \tag{2.11}$$

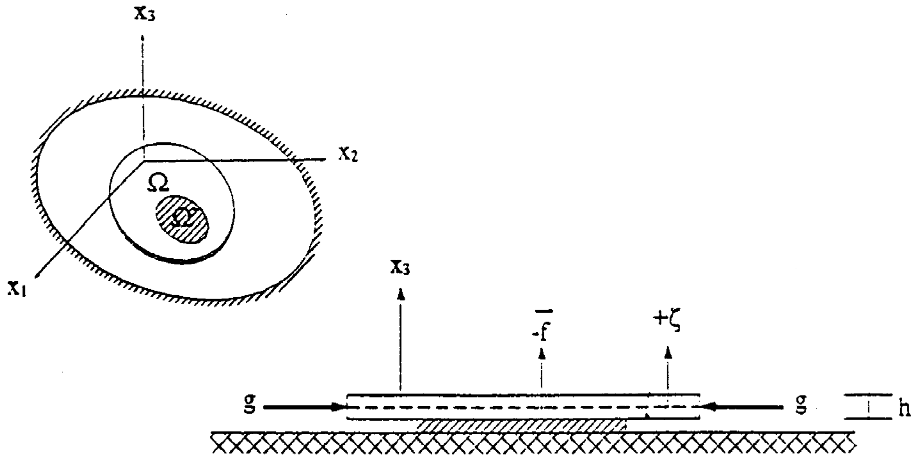

Figure 1. Geometry and notation of the mechanical model studied.

where g_a are selfequilibrating forces and λ is a scalar parameter.

For the moment we assume that $g_a = 0, \ = 0,\ a = 1, 2.$

Let us now introduce the functional framework within which the previously formulated B.V.P will be studied We assume that $u,\ v \in [\overset{\circ}{H}{}^1(\Omega)]^2$ and $\zeta,\ z \in Z$, where

$$Z = \{z/z \in H^2(\Omega), z = 0 \text{ on } \Gamma\}\,.$$

or $Z = H^2(\Omega) \cap \overset{\circ}{H}{}^1(\Omega)$.

Moreover, let $f \in L^2(\Omega),\ \Delta\Delta\zeta \in L^2(\Omega)$.

We will now derive the variational formulation of the problem. From (2.1), multiplying by $z - \zeta$, integrating and applying the Green–Gauss theorem (cf. Panagiotopoulos, 1985b, Section 7.1.1), we obtain the expressions

$$\begin{aligned} a(\zeta, z) &+ \int_\Omega h\sigma_{a\beta}\zeta_{,a}(z-\zeta)_{,\beta}\,\mathrm{d}\Omega \\ &= \int_\Gamma h\sigma_{a\beta}\zeta_{,a}n_\beta(z-\zeta)\,\mathrm{d}\Gamma + \int_\Gamma Q(\zeta)(z-\zeta)\,\mathrm{d}\Gamma \\ &\quad - \int_\Gamma M(\zeta)\frac{\partial(z-\zeta)}{\partial n}\,\mathrm{d}\Gamma + \int_\Omega f(z-\zeta)\,\mathrm{d}\Omega\,, \end{aligned} \tag{2.12}$$

where $a, \beta = 1, 2$.

Here

$$a(\zeta, z) = K\int_\Omega [(1-\nu)\zeta_{,a\beta}z_{,a\beta} + \nu\,\Delta\zeta\,\Delta z]\,\mathrm{d}\Omega,\ a,\ \beta = 1, 2,\ <\nu< 1/2 \tag{2.13}$$

is the bilinear form of the elastic energy of the *von Kármán* plate,

$$M(\zeta) = -K[\nu\,\Delta\zeta + (1-\nu)(2n_1n_2\zeta_{,12} + n_1^2\zeta_{,11} + n_2^2\zeta_{,22})] \tag{2.14}$$

is the bending moment and

$$Q(\zeta) = -K\left[\frac{\partial\,\Delta\zeta}{\partial n} + (1-\nu)\frac{\partial}{\partial\tau}[n_1 n_2(\zeta_{,22} - \zeta_{,11}) + (n_1^2 - n_2^2)\zeta_{,12}]\right] \tag{2.15}$$

is the total shearing force on the plate boundary.

Here ν is the Poisson ratio and τ is the unit vector tangential to Γ, such that ν, τ and the Ox_3 axis form a right-handed system.

Applying the same technique of variational calculus to the in-plane relation (2.2), we have that

$$\int_\Omega \sigma_{\alpha\beta}\varepsilon_{\alpha\beta}(v-u)\,\mathrm{d}\Omega = \int_\Gamma \sigma_{\alpha\beta}n_\beta(v_a - u_a)\,\mathrm{d}\Gamma\,. \tag{2.16}$$

Equations (2.12) and (2.16) express the principle of virtual work for the bending and the stretching respectively of the plate.

We introduce the following notations

$$R(h,k) = \int_\Omega C_{\alpha\beta\gamma\delta}h_{\alpha\beta}k_{\gamma\delta}\,\mathrm{d}\Omega, \quad \alpha,\beta,\gamma,\delta = 1,2 \tag{2.17}$$

and

$$P(\zeta, z) = \{\zeta_{,a}z_{,\beta}\},\ P(\zeta,\zeta) = P(\zeta)\,, \tag{2.18}$$

where $h = \{h_{\alpha\beta}\}$ and $k = \{k_{\alpha\beta}\}$, $\alpha, \beta = 1, 2$ are 2×2 tensors.

By taking into account the variational equalities (2.12), (2.16), the boundary conditions and the inequality giving the multivalued law (2.6)

$$j^0(\zeta, z-\zeta) \geqslant \bar{f}(z-\zeta)\,, \tag{2.19}$$

we obtain the variational formulation of the delamination problem of one *von Kármán* plate adhesively connected to a rigid support:

(P) find $u \in [H^1(\Omega)]^2$ and $\zeta \in Z$, such as to satisfy the hemivariational inequality

$$a(\zeta, z-\zeta) + hR(\varepsilon(u) + \frac{1}{2}P(\zeta), P(\zeta, z-\zeta)) + \int_{\Omega'} j^0(\zeta, z-\zeta)\,\mathrm{d}\Omega$$
$$\geqslant \int_\Omega \bar{\bar{f}}(z-\zeta)\,\mathrm{d}\Omega\,, \quad \forall z \in Z \tag{2.20}$$

and the variational equality

$$R(\varepsilon(u) + \frac{1}{2}P(\zeta), \varepsilon(v-u)) = 0\,, \quad \forall v \in [H^1(\Omega)]^2\,. \tag{2.21}$$

Further we shall eliminate the in-plate displacements of the plate. To this end we note first that $R(.\,,.)$ as defined in (2.17) is a continuous, symmetric, coercive bilinear form on $[L^2(\Omega)]^4$ and that $P: H^2(\Omega) \to [L^2(\Omega)]^4$ of (2.18) is a completely continuous operator. Thus (2.21) and the Lax–Milgram theorem imply that to every deflection $\zeta \in Z$, there corresponds a plane displacement $u(\zeta) \in [H^1(\Omega)]^2$. Indeed, due to Korn's inequality, $R(\varepsilon(u), \varepsilon(v))$ is a bilinear coercive form on the quotient space $[H^1(\Omega)]^2/\overline{R}$, where $\overline{R}$ is the space of the in-plane rigid-plate displacements defined by

$$\overline{R} = \{\bar{r}/\bar{r} \in [H^1(\Omega)]^2, \bar{r}_1 = a_1 + bx_2, \bar{r}_2 = a_2 - bx_1, a_1, a_2, b \in R\} . \tag{2.22}$$

From (2.21) it results that

$$\varepsilon(u(\zeta)) : Z \to [L^2(\Omega)]^4 \tag{2.23}$$

is uniquely determined and is a completely continuous quadratic function of ζ, since $\varepsilon(u(\zeta))$ is a linear continuous function of $P(\zeta)$. We also introduce the completely continuous quadratic function $G : Z \to [L^2(\Omega)]^4$ which is defined by

$$\zeta \mapsto G(\zeta) = \varepsilon(u(\zeta)) + \frac{1}{2} P(\zeta) \tag{2.24}$$

and satisfies the equation

$$R(G(\zeta), \varepsilon(u(\zeta))) = 0 . \tag{2.25}$$

We define the operators $A : Z \to Z'$ and $C : Z \to Z'$ (where Z' is the dual of Z and $\langle . , . \rangle_Z$ the duality pairing between Z and Z'), such that

$$\langle A\zeta, z\rangle_Z = a(\zeta, z) \tag{2.26}$$

and

$$\langle C(\zeta), z\rangle_Z = hR(G(\zeta), P(\zeta, z)) . \tag{2.27}$$

From (2.25) and (2.27), it results that

$$\langle C(\zeta), \zeta\rangle_Z = hR(G(\zeta), 2G(\zeta)) \geqslant 0 . \tag{2.28}$$

Thus problem (P) results to the following form:

find $\zeta \in Z$, so as to satisfy the hemivariational inequality

$$a(\zeta, z - \zeta) + \langle C(\zeta), z - \zeta\rangle_Z + \int_{\Omega'} j^0(\zeta, z - \zeta)\, d\Omega \geqslant \int_{\Omega} \bar{\bar{f}}(z - \zeta)\, d\Omega , \quad \forall z \in Z . \tag{2.29}$$

The last hemivariational inequality characterizes the position of equilibrium of the delamination problem.

We are now in the position to formulate the corresponding eigenvalue problem. To this end we apply the method presented in Panagiotopoulos (1985b), Section 7.2, by assuming that (2.11) holds with g_1 and g_2 nonzero. We obtain the following eigenvalue problem for a hemivariational inequality:

(P_λ) find $(\zeta, \lambda) \in Z \times R$ such as to satisfy the expression

$$a(\zeta, z-\zeta) + \langle C(\zeta), z-\zeta\rangle_Z + \int_{\Omega'} j^0(\zeta, z-\zeta)\,\mathrm{d}\Omega$$
$$\geqslant \lambda(B\zeta, z-\zeta)_Z\,, \quad \forall z \in Z\,, \tag{2.30}$$

where $(.\,,.)_Z$ is the inner product of Z and B is a linear selfadjoint compact operator such that

$$(B\zeta, z)_Z = -h\int_\Omega \sigma^0_{\alpha\beta}\zeta_{,\alpha}z_{,\beta}\,\mathrm{d}\Omega\,, \quad \forall \zeta, z \in Z\,. \tag{2.31}$$

For the meaning of $\sigma^0_{\alpha\beta}$ see Panagiotopoulos (1985b), Section 7.2.

The eigenvalue problem (2.31) will be studied in the next sections on the technical assumption (cf. Panagiotopoulos, 1985b, Equation 7.2.64) that the boundary Γ_1 of every subdomain $\Omega_1 \subset \Omega$ of the plate is subjected to compressive forces, i.e., that almost everywhere on Γ_1 the inequality

$$\sigma^0\bar{n}_i\bar{n}_j \leqslant 0 \tag{2.32}$$

holds, where $\bar{n} = \{\bar{n}_i\}$ is the unit normal outward vector on Γ_1. Then (2.32) implies that

$$(B\zeta, \zeta)_Z > 0\,, \quad \forall \zeta \in Z,\ \zeta \neq 0\,. \tag{2.33}$$

For general unilateral problems of *von Kármán* plates, the variational inequality formulation is not equivalent to a minimum problem because of the geometric nonlinearity of the physical model. However, problem (P_λ) is equivalent to the following problem (cf. Panagiotopoulos, 1985b, Proposition 7.1.3):

find $\zeta \in Z$ such that for some $\lambda \in R$,

$$0 \in \bar{\partial} I(\zeta)\,, \tag{2.34}$$

where

$$I(\zeta) = \frac{1}{2}a(\zeta, \zeta) + \frac{h}{2}R(G(\zeta), P(\zeta)) + \int_{\Omega'} j(\zeta)\,\mathrm{d}\Omega - \frac{\lambda}{2}(B\zeta, \zeta)z\,, \tag{2.35}$$

corresponds to the potential energy of the plate.

The equivalency of the solutions of the foregoing problem with problem (P_λ) is discussed in Panagiotopoulos (1985b, pp. 150–151.

We consider now the sphere S_r in Z, described as follows

$$S_r = \{\zeta \in Z : (B\zeta, \zeta)_Z = r^2\}\,,$$

where r is some positive number. In mathematical terms S_r is a submanifold of Z. Corresponding to each $r > 0$ we formulate problem (P_λ) in the following form:

$(P_{\lambda,r})$ find $\zeta \in S_r$ and $\lambda \in R$ such that

$$a(\zeta, z-\zeta) + \langle C(\zeta), z-\zeta\rangle_Z + \int_{\Omega'} j^0(\zeta, z-\zeta)\, \mathrm{d}\Omega \geqslant \lambda(B\zeta, z-\zeta)_Z\,, \quad \forall z \in Z\,. \tag{2.36}$$

Problem $(P_{\lambda,r})$ corresponds to the buckling of the considered *von Kármán* plate for given cost or weight. The constraint $(B\zeta, \zeta)_Z = r^2$ imposed, means that we have a system with prescribed cost or weight or, in some cases energy consumption.

3. The mathematical study

3.1. AN EXISTENCE RESULT

We consider space Z to be equipped with the classical H^2-norm. Z being a closed subspace of $H^2(\Omega)$, is a Banach space $(Z, \|\cdot\|_Z)$. Z is also a Hilbert space with the inner product $(.\,,.)_Z$ associated to the norm $\|\cdot\|_Z$. We denote by $\langle.\,,.\rangle_Z$ the duality pairing between Z and Z'.

Z is densely and compactly imbedded in $L^2(\Omega)$, where Ω is a bounded connected domain of R^2. Let us denote by $C_2(\Omega)$ the positive constant of the imbedding $Z \subset L^2(\Omega)$, which means that

$$\|\zeta\|_{L^2} \leqslant C_2(\Omega)\|\zeta\|_Z\,, \quad \forall \zeta \in Z\,. \tag{3.1.1}$$

We remind that $a: Z \times Z \to R$ is a continuous symmetric bilinear form, $B: Z \to Z$ is a linear compact operator and $C: Z \to Z'$ is a (nonlinear) compact operator.

Let the function $\beta \in L^\infty_{\mathrm{loc}}(R)$ satisfy the assumption

(H_1) there exist constants $a_1, a_2 \in R$ with $a_2 > 0$ such that the following growth condition holds

$$|\beta(t)| \leqslant a_1 + a_2|t|\,, \quad \forall \in R\,, \tag{3.1.2}$$

where $|\cdot|$ is the Euclidean norm.

We define now the function $j: L^2(\Omega) \to R$

$$j(t) = \int_0^t \beta(s)\, \mathrm{d}s\,, \quad t \in R\,. \tag{3.1.3}$$

An existence result for (P_λ) is stated in the following theorem.

THEOREM 1. *Assume that the hypotheses* (H_1) *holds. Then, for every* $\lambda \in R$ *satisfying*

$$-\|a\| - a_2(C_2(\Omega))^2 - \lambda\|B\| > 0\,, \tag{3.1.4}$$

there exists $\zeta \in Z$ *solving* (P_λ).

Proof. Let us fix some $\lambda \in R$ as in the statement of the theorem. Corresponding to this λ let $I : Z \to R$ be the functional defined by

$$(I\zeta) = \frac{1}{2} a(\zeta, \zeta) + \frac{h}{2} R(G(\zeta), P(\zeta)) + \int_{\Omega'} j(\zeta)\, d\Omega - \frac{\lambda}{2} (B\zeta, \zeta)_Z , \quad \forall \zeta \in Z . \tag{3.1.5}$$

The functional I is well defined and locally Lipschitz.

We also define the functional $J : L^2(\Omega) \to R$ such that

$$J(\zeta) = \int_{\Omega'} j(\zeta)\, d\Omega , \quad \zeta \in L^2(\Omega) . \tag{3.1.6}$$

In view of (H_1), J is well defined and locally Lipschitz in Z (cf. Chang, 1981). We denote by $J|_Z$ the restriction of J in Z.

We will prove that the functional I is bounded from below on Z. Indeed from (2.28), 3.1.1) and hypothesis (H_1), we obtain the estimate

$$\begin{aligned} I(\zeta) &\geqslant -\frac{1}{2}\|a\|\,\|\zeta\|_Z^2 - a_1(\text{meas}(\Omega))^{1/2}\|\zeta\|_{L^2} - \frac{1}{2} a_2\|\zeta\|_{L^2}^2 - \frac{\lambda}{2}\|B\|\,\|\zeta\|_Z^2 \\ &\geqslant -\frac{1}{2}\|a\|\,\|\zeta\|_Z^2 - a_1(\text{meas}(\Omega))^{1/2} C_2(\Omega)\|\zeta\|_Z - \frac{1}{2} a_2(C_2(\Omega))^2\|\zeta\|_Z^2 \\ &\quad -\frac{\lambda}{2}\|B\|\,\|\zeta\|_Z^2 \end{aligned}$$

or

$$\begin{aligned} I(\zeta) &\geqslant \Big(-\frac{1}{2}\|a\| - \frac{1}{2} a_2(C_2(\Omega))^2 - \frac{\lambda}{2}\|B\|\Big)\|\zeta\|_Z^2 \\ &\quad - a_1(\text{meas}(\Omega))^{1/2} C_2(\Omega)\|\zeta\|_Z , \quad \forall \zeta \in Z , \end{aligned} \tag{3.1.7}$$

where meas(Ω) is the Lebesgue measure of Ω.

Hence for the given λ the functional I is bounded from below on Z, i.e., there exists $c = \inf_Z I > -\infty$.

Taking into account (3.1.5) and employing the calculus with generalized gradients, it turns out that

$$\bar{\partial} I(\zeta) = A\zeta + \bar{\partial}(J|_Z)(\zeta) - \lambda \Lambda B\zeta + C(\zeta) , \tag{3.1.8}$$

where $\Lambda : Z \to Z'$ is the duality mapping $\langle \Lambda\zeta, z\rangle_Z = (\zeta, z)_Z$, $\zeta, z \in Z$.

We will prove now that the locally Lipschitz functional I satisfies the Palais–Smale condition in the sense of Chang (see, e.g., Panagiotopoulos, 1993, p. 180). Accordingly, let a sequence $(\zeta_n) \subset Z$ fulfill

$$I(\zeta_n) \leqslant M , \tag{3.1.9}$$

for constant $M > 0$ and for some $J_n \in \bar{\partial} I(\zeta_n)$ with

$$J_n \to 0 \text{ in } Z' \text{ as } n \to \infty . \tag{3.1.10}$$

From (3.1.7), (3.1.9) one finds easily that $I(\zeta_n)$ is bounded on Z. The boundedness of I and the fact that I is locally Lipschitz lead to the conclusion that (ζ_n) is bounded in Z.

In view of (3.1.8), (3.1.10), let a sequence $\omega_n \in \bar{\partial}(J|_Z)(\zeta_n)$ such that

$$J_n = A\zeta_n + \omega_n - \lambda\Lambda B\zeta_n + C(\zeta_n) \to 0 \text{ in } Z' \text{ as } n \to \infty\,. \tag{3.1.11}$$

The boundedness of (ζ_n) in Z and the compactness of the mapping $C: Z \to Z'$ implies the convergence of $C(\zeta_n)$ along a subsequence in Z denoted again by $C(\zeta_n)$. The compactness of the imbedding $Z \subset L^2(\Omega)$ assures that a subsequence of (ζ_n) also denoted by (ζ_n), converges in $L^2(\Omega)$. On the other hand the density of the imbedding $Z \subset L^2(\Omega)$ implies that

$$\omega_n \in \bar{\partial}(J|_Z)(\zeta_n) \subset \bar{\partial}J(\zeta_n)\,, \quad \forall n\,, \tag{3.1.12}$$

(according to Theorem 2.2 of Chang, 1981).

The facts that J is locally Lipschitz on $L^2(\Omega)$ and that (ζ_n) is bounded in Z ensure that (ω_n) is bounded in $L^2(\Omega)$. By the compactness of the imbedding $L^2(\Omega) \subset Z'$ it turns out that (ω_n) converges along a subsequence in Z'. Letting $n \to \infty$ in (3.1.11) implies the convergence along a subsequence of $(-A\zeta_n + \lambda\Lambda B\zeta_n)$ in Z'.

Notice that (3.1.4) implies

$$-\|a\| - \lambda\|B\| > 0\,. \tag{3.1.13}$$

Now we can write the following inequality

$$\begin{aligned}(-\|a\| - \lambda\|B\|)\|\zeta_n - \zeta_m\|_Z^2 &\leqslant a(\zeta_n - \zeta_m, \zeta_n - \zeta_m) - \lambda(B(\zeta_n - \zeta_m), \zeta_n - \zeta_m)_Z \\ &\leqslant \|A(\zeta_n - \zeta_m) - \lambda\Lambda B(\zeta_n - \zeta_m)\|_{Z'}\|\zeta_n - \zeta_m\|_Z\,, \quad \forall m, n \in N, m, n \geqslant 1\,.\end{aligned} \tag{3.1.14}$$

The convergence $(-A\zeta_n + \lambda\Lambda B\zeta_n)$ in Z' and the relations (3.1.13) and (3.1.14) show that (ζ_n) contains a Cauchy subsequence in Z'; thus (ζ_n) converges along a subsequence in Z to ζ. Hence the Palais–Smale condition for the functional I is true.

The boundedness property of I and the Palais–Smale condition for I that was just verified, are the only requirements to apply the Palais–Smale minimization theorem in Chang's version of the locally Lipschitz functions to the functional I (see, e.g., Panagiotopoulos, 1993, p. 180). It follows that there exists $\zeta \in Z$ with

$$I(\zeta) = \inf_Z I\,. \tag{3.1.15}$$

In particular, we derive that

$$0_Z \in \bar{\partial}I(\zeta) \tag{3.1.16}$$

or by taking into account (3.1.8)

$$a(\zeta, z) - \lambda(B\zeta, z)_Z + J^0(\zeta; z) + hR(G(\zeta), P(\zeta, z)) \geqslant 0\,. \tag{3.1.17}$$

Given that the functional J is locally Lipschitz in Z, its generalized gradient satisfies

$$\bar{\partial} J(\zeta) \subset \int_\Omega \bar{\partial} j(\zeta)\, d\Omega\,, \quad \forall \zeta \in Z \tag{3.1.18}$$

(cf. Panagiotopoulos, 1993, Proposition 2.5.3).

From (3.1.12), (3.1.18) it results that

$$\bar{\partial}(J|_Z)(\zeta) \subset \int_\Omega \bar{\partial} j(\zeta)\, d\Omega\,, \quad \forall \zeta \in Z\,. \tag{3.1.19}$$

From (3.1.17), (3.1.19) we conclude that

$$-a(\zeta, z) + \lambda(B\zeta, z)_Z - hR(G(\zeta), P(\zeta, z)) \leqslant \int_\Omega \max_{q \in \bar{\partial} j(z)} q(z)\, d\Omega \tag{3.1.20}$$

and by recalling the definition of the generalized gradient

$$-a(\zeta, z) + \lambda(B\zeta, z)_Z - hR(G(\zeta), P(\zeta, z)) \leqslant \int_\Omega j^0(\zeta; z)\, d\Omega\,, \quad \forall z \in Z\,. \tag{3.1.21}$$

This completes the proof of Theorem 1. □

3.2. A MULTIPLICITY RESULT

In this section we assume that Z is equipped with the $(B\zeta, \zeta)^{1/2}$-norm, in contrast to Section 3.1 where Z is equipped with the classical H^2-norm. Moreover the duality mapping $\Lambda : Z \to Z'$ is given by

$$\langle \Lambda\zeta, z\rangle_Z = (B\zeta, z)_Z\,. \tag{3.2.1}$$

We make the following hypotheses

(H_2) for every $\zeta \in Z$, $j(\cdot)$ is even.

(H_3) for every sequence $(\zeta_n) \in S_r$ with $\zeta_n \to \zeta$ weakly in Z,

$$a(\zeta_n, \zeta_n) + \langle C(\zeta_n), \zeta_n\rangle_Z \to a_0 \in R \tag{3.2.2}$$

and for every $x \in Z'$ with

$$x \in \bar{\partial} j(\zeta)\,, \tag{3.2.3}$$

then

$$\|a\| - \frac{1}{r^2}\left(a_0 + \int_\Omega \langle x, \zeta\rangle_Z\, d\Omega\right) > 0\,. \tag{3.2.4}$$

The theoretical basis of the following theorem is Theorem 3.2 of Chang (1981).

THEOREM 2. *Assume that hypotheses* (H_1), (H_2), (H_3) *hold. Then, problem* $(P_{\lambda,r})$ *admits infinitely many distinct pairs of solutions* $(\pm\zeta_n, \lambda_n)_{n\geqslant 1} \subset S_r \times R$ *with*

$$\lambda_n = \frac{1}{r^2}\left(\langle A(\zeta_n) + C(\zeta_n),\, \zeta_n\rangle_Z + \int_\Omega \langle x_n,\, \zeta_n\rangle_Z \, d\Omega\right), \quad n \geqslant 1\,, \tag{3.2.5}$$

where $x_n \in \bar{\partial} j(\pm \zeta_n)$.

Proof. We consider the functional $F : Z \to R$ given by

$$F(\zeta) = \frac{1}{2}a(\zeta,\zeta) + \frac{h}{2}R(G(\zeta), P(\zeta)) + J|_Z(\zeta)\,, \tag{3.2.6}$$

where $J : L^2(\Omega) \to R$ the functional

$$J(\zeta) = \int_\Omega j(\zeta)\, d\Omega\,, \quad \zeta \in L^2(\Omega)\,. \tag{3.2.7}$$

and $J|_Z$ the restriction of J and Z. In view of hypothesis (H_1), J is a locally Lipschitz functional on $L^2(\Omega)$ or Z.

We check now that the locally Lipschitz functional F is even. We have that

$$a(-\zeta, -\zeta) = a(\zeta, \zeta)\,, \tag{3.2.8}$$

and

$$R(G(-\zeta), P(-\zeta)) = R(G(\zeta), P(\zeta)), \quad \text{as} \quad G(-\zeta) = G(\zeta), P(-\zeta) = P(\zeta)\,. \tag{3.2.9}$$

From (3.2.8), (3.2.9) and hypotheses (H_2), it follows that F is even, i.e.,

$$F(-\zeta) = F(\zeta) \text{ for every } \zeta \in Z\,. \tag{3.2.10}$$

Let $F|_{S_r}$ the restriction of F on the sphere S_r.

From (2.28), (3.1.1) and hypothesis (H_1) we see that $F|_{S_r}$ is bounded from below:

$$\begin{aligned}(F|_{S_r})(\zeta) &\geqslant -\frac{1}{2}\|a\|\,\|\zeta\|_Z^2 - a_1(\text{meas}(\Omega))^{1/2}\|\zeta\|_{L^2} - \frac{1}{2}a_2\|\zeta\|_{L^2}^2 \\ &\geqslant -\frac{1}{2}\|a\|\,\|\zeta\|_Z^2 - a_1(\text{meas}(\Omega))^{1/2}C_2(\Omega)\|\zeta\|_Z - \frac{1}{2}a_2(C_2(\Omega))^2\|\zeta\|_Z^2 \\ &\geqslant \left(-\frac{1}{2}\|a\| - \frac{1}{2}a_2(C_2(\Omega))^2\right)r^2 - a_1(\text{meas}(\Omega))^{1/2}C_2(\Omega)r\,,\end{aligned} \tag{3.2.11}$$

where meas(Ω) is the Lebesgue measure of Ω.

For continuing the proof, it is necessary to remark that the expression of the generalized gradient $\bar{\partial}(F|_{S_r})(\zeta)$ at the point $\zeta \in S_r$ is given by the formula

$$\bar{\partial}(F|_{S_r})(\zeta) = \left\{q - \frac{1}{r^2}\langle q, \zeta\rangle_Z \Lambda\zeta : q \in \bar{\partial}F(\zeta)\,, \quad \forall \zeta \in S_r\right\}, \tag{3.2.12}$$

where $\bar{\partial}F(\zeta) = A\zeta + C(\zeta) + \bar{\partial}(J|_Z)(\zeta)$.

The next step is to prove that the functional $F|_{S_r}$ satisfies the Palais–Smale condition in the sense of Chang. Let us consider a sequence $(\zeta_n) \in S_r$ such that

$(F|_{S_r})(\zeta_n) \leqslant M$ for constant $M > 0$ and such that there exists some sequence $(q_n) \in Z'$ fulfilling the conditions

$$q_n \in \bar{\partial} F(\zeta_n) \tag{3.2.13}$$

and

$$q_n - \frac{1}{r^2} \langle q_n, z_n \rangle_Z \Lambda \zeta_n \to 0, \text{ in } Z' \text{ as } n \to \infty , \tag{3.2.14}$$

or equivalently

$$A\zeta_n + C(\zeta_n) + w_n - \frac{1}{r^2} \langle A\zeta_n + C(\zeta_n) + w_n,\ \zeta_n \rangle_Z \Lambda \zeta_n \to 0$$
$$\text{in } Z' \text{ as } n \to \infty , \tag{3.2.15}$$

where $w_n \bar{\partial}(J|_Z)(\zeta_n), \forall n$.

We have to prove that (ζ_n) contains a convergent subsequence in Z.

By the fact that the sequence (ζ_n) is contained in S_r it is obvious that (ζ_n) is bounded in Z. So up to a subsequence we may conclude that

$$\zeta_n \to \zeta \text{ weakly in } Z \text{ as } n \to \infty \text{ for some } \zeta \in Z . \tag{3.2.16}$$

The compactness of the imbedding $Z \subset L^2(\Omega)$, provides the convergence

$$\zeta_n \to \zeta \text{ in } L^2(\Omega) . \tag{3.2.17}$$

The density of the imbedding $Z \subset L^2(\Omega)$ and theorem 2.2 of Chang (1981) imply that

$$\omega_n \in \bar{\partial}(J|_Z)(\zeta_n) \subset \bar{\partial} J(\zeta_n) . \tag{3.2.18}$$

Since J is locally Lipschitz on $L^2(\Omega)$ and (ζ_n) is bounded in Z, (ω_n) is also bounded in $L^2(\Omega)$. Thus, for a subsequence of (ω_n) also denoted by (ω_n), we have that

$$\omega_n \to \omega \text{ weakly in } L^2(\Omega) \text{ as } n \to \infty \tag{3.2.19}$$

and due to the compactness of the imbedding $L^2(\Omega) \subset Z'$,

$$\omega_n \to \omega \text{ in } Z' \text{ as } n \to \infty . \tag{3.2.20}$$

Taking into account (3.2.16) and (3.2.20)

$$\langle \omega_n, \zeta_n \rangle_Z \to \langle \omega, \zeta \rangle_Z, \text{ as } n \to \infty . \tag{3.2.21}$$

By the boundedness of ζ_n in Z one supposes that $a(\zeta_n, \zeta_n)$ converges in R. The boundedness of (ζ_n) and the compactness of the mapping $C: Z \to Z'$ imply the convergence of $C(\zeta_n)$ along a subsequence in Z'. Thus knowing that ζ_n converges weakly in Z,

$$a(\zeta_n, \zeta_n) + \langle C(\zeta_n), \zeta_n \rangle_Z \to a0 \in R . \tag{3.2.22}$$

By taking into account the convergences stated above, from (3.2.15) we derive that

$$A\zeta_n - \frac{1}{r^2}(a_0 + \langle \omega, \zeta \rangle_Z)\Lambda\zeta_n \text{ converges in } Z' \text{ as } n \to \infty . \tag{3.2.23}$$

From (3.2.17), (3.2.20) we conclude that

$$\omega \in \bar{\partial} J(\zeta) \subset \int_\Omega \bar{\partial} j(\zeta)\, \mathrm{d}\Omega \tag{3.2.24}$$

and consequently

$$\langle w, \zeta \rangle_Z = \int_\Omega \langle x, \zeta \rangle_Z \, \mathrm{d}\Omega , \tag{3.2.25}$$

where $x \in \bar{\partial} j(\zeta)$.

Now we can write the following inequality

$$\begin{aligned}
&\left(\|a\| - \frac{1}{r^2}\left(a_0 + \int_\Omega \langle x, \zeta \rangle_Z \, \mathrm{d}\Omega \right)\right) \|\zeta_n - \zeta_m\|_Z^2 \\
&\quad \leqslant a(\zeta_n - \zeta_m, \zeta_n - \zeta_m) - \frac{1}{r^2}\left(\left(a_0 + \int_\Omega \langle x, \zeta \rangle_Z \, \mathrm{d}\Omega \right)(\zeta_n - \zeta_m), \zeta_n - \zeta_m \right)_Z \\
&\quad \leqslant \left\| A(\zeta_n - \zeta_m) - \frac{1}{r^2}\left(a_0 + \int_\Omega \langle x, \zeta \rangle_Z \, \mathrm{d}\Omega \right)\Lambda(\zeta_n - \zeta_m) \right\|_{Z'} \|\zeta_n - \zeta_m\|_Z , \\
&\forall m, n \in N, m, n \geqslant 1 .
\end{aligned} \tag{3.2.26}$$

The relations (3.2.4), (3.2.23) and hypothesis (H_3) show that (ζ_n) contains a Cauchy sequence in Z; thus (ζ_n) converges along a subsequence in Z to ζ. This completes the verification fo the Palais–Smale condition for $F|_{S_r}$.

At this point, we have verified all the requirements for Theorem 3.2 of Chang (1981) for locally Lipschitz functions.

Let us denote by Y the family of closed and symmetric with respect to the origin 0_Z, subsets of S_r. Let us denote by $\gamma(S)$ the Krasnoselski genus of the set $S \in Y$, that is the smallest integer $k \in N \cup \{+\infty\}$ for which there is an odd continuous mapping from S into $R^k \backslash \{0\}$. For every $n \geqslant 1$, let

$$\Gamma_n = \{ S \subset S_r : S \in Y, \gamma(S) \geqslant n \} . \tag{3.2.27}$$

According to Chang's theorem, the critical points of F on S_r are

$$\beta_n = \inf_{S \subset \Gamma_n} \sup_{\zeta \in S} F|_{S_r}(\zeta) , \tag{3.2.28}$$

i.e., there exist critical points $(\pm \zeta_n)$ such that

$$0_Z \in (\bar{\partial} F|_{S_r})(\pm \zeta_n) , \tag{3.2.29}$$

with

$$(F|_{S_r})(\pm \zeta_n) = \beta_n , \quad n \geqslant 1 . \tag{3.2.30}$$

From the formula (3.2.29) it follows that there exist $q_n \in \bar{\partial} F(\pm \zeta_n)$ such that

$$q_n - \frac{1}{r^2}\langle q_n, \zeta_n\rangle_Z \Lambda(\pm\zeta_n) = 0\,. \tag{3.2.31}$$

Having

$$\begin{aligned}\langle q_n, z\rangle_Z &\in \bar{\partial} F(\pm\zeta_n)z \\ &\subset \langle A(\pm\zeta_n), z\rangle_Z + \langle C(\pm\zeta_n), z\rangle_Z + \int_\Omega \bar{\partial} j(\pm\zeta_n) z \, \mathrm{d}\Omega\,,\end{aligned} \tag{3.2.32}$$

for $x_n \in \bar{\partial} j(\pm\zeta_n)$, (3.2.32) is equivalent to

$$\langle q_n, z\rangle_Z = a(\pm\zeta_n, z) + \langle C(\pm\zeta_n), z\rangle, z\rangle_Z + \int_\Omega \langle x_n, z\rangle_Z \, \mathrm{d}\Omega\,. \tag{3.2.33}$$

From (3.2.31), (3.2.33) it results that

$$\begin{aligned}&a(\pm\zeta_n, z) + \langle C(\pm\zeta_n), z\rangle_Z + \int_\Omega \langle x_n, z\rangle_Z \, \mathrm{d}\Omega \\ &\quad - \frac{1}{r^2}\Big(a(\pm\zeta_n, \pm\zeta_n) + \langle C(\pm\zeta_n), \pm\zeta_n\rangle_Z \\ &\quad + \int_\Omega \langle x_n, \pm\zeta_n\rangle_Z \, \mathrm{d}\Omega\Big)(B(\pm\zeta_n), z)_Z = 0\,.\end{aligned} \tag{3.2.34}$$

We remind that $x_n \in \bar{\partial} j(\pm\zeta_n)$ means that

$$\langle x_n, \pm\zeta_n\rangle_Z \leqslant j^0(\pm\zeta_n, z)\,, \tag{3.2.25}$$

which makes (3.2.24) as follows

$$\begin{aligned}&a(\pm\zeta_n, z) + \langle C(\pm\zeta_n), z\rangle_Z + \int_\Omega j^0(\pm\zeta_n, z) \, \mathrm{d}\Omega \\ &\quad \geqslant \lambda_n (B(\pm\zeta_n, z)_Z \quad \forall \zeta \in Z\,.\end{aligned} \tag{3.2.36}$$

This completes the proof of Theorem 2. The mechanical interpretation of the statement proved, is that there exist infinitely many equilibrium states of the adhesively supported *von Kármán* plate. □

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